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SOME NORTH PACIFIC SPECIES OF ARTICULATED CORALLINES

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In an earlier paper (Manza, "The Genera of the Articulated Corallines," *Proc. Nat. Acad. Sci.*, **23**, No. 2, 44–48, 1937) I have indicated certain lines of demarcation between the different genera of the Articulated Corallines. In attempting to arrange species hitherto described or undescribed under the genera thus instituted, it has become necessary to propose certain new species and also certain new specific combinations. The present paper deals with some of those occurring in the North Pacific area.

1. *Bossea californica* (Decne.) comb. nov.—*Amphiroa* (*Arthrocardia*) *californica* Decaisne, Mem. sur les Corallines, pp. 112, 1842.

Fronds 5–12 cm. long; branching dichotomous-digitate-corymboid; intergenicula near the base thick cylindrical, 1–3 mm. long and 2 mm. broad, and those of the upper parts thick compressed, mostly obcordate, with lobes rounded, 2–5 mm. long and 2–6 mm. broad; tetrasporic conceptacles on the flat surfaces of the intergenicula, 2–8, mostly 6, on each surface, arranged in rows along the lateral margins, 2–4 on each side, with pores central.

Species type: *Amphiroa* (*Arthrocardia*) *californica* Decaisne (loc. cit., p. 112).

Type locality: "California (Monterey)."

Species topotype: tetrasporic, Herb. Univ. Calif. No. 266290; Pacific Grove, Monterey County, coast of central California, west coast of North America. Coll. N. L. Gardner.

Synonym: *Cheilosporum californicum* Yendo, a revised list of Corallinae. Jour. Coll. Sci. Imp. Univ. Tokyo, Japan, 20, art. 12:19, 1905.

The type of the species is not available, but topotypes have been examined. According to our generic interpretations, this species undoubtedly belongs to *Bossea* and not to any one of the genera to which it has hitherto been referred. *Bossea californica* is a larger species than *Bossea corymbifera*, and, in addition, the former has mostly 6 conceptacles on each surface

of the intergeniculum while the latter has most generally only 2 on each surface of the intergeniculum.

2. *Bossea corymbifera* sp. nov.—B. frondibus 3–5 cm. longis; ramificatione bi-tri-pinnatocorymboidea; intergeniculis subbasalibus crassis cylindricis, 1–2 mm. longis, 1 mm. latis, iis partium superarum compressis, crasso-cuneatis aut obcordatis, lobis obtusis rotundisve, 1–2 mm. longis, 2–3 mm. latis; conceptaculis tetrasporangialibus semiglobosis in superficiebus planis intergeniculorum positis, in superficiebus quibusque 2–4 singulis aut binis et in seriebus verticalibus per axes dispositis, poris apicalibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 545752; Point Lobos, Carmel Bay, coast of central California, West coast of North America. Coll. A. V. Manza.

This species has thick basal segments and also has the branches of the lower parts growing faster than those of the upper parts, thus giving the fronds a corymboid appearance.

3. *Bossea dichotoma* sp. nov.—B. frondibus 6–12 cm. longis; ramificatione primaria pinnata subalternaque, ultime dichotoma; intergeniculis subbasalibus tenuibus compressisque, 1–2 mm. longis, 1.5 mm. latis, iis partium superarum compressis, cuneatis aut subcordatis, 2–3 mm. longis, 2–6 mm. latis, lobis rotundatis; conceptaculis tetrasporangialibus semiglobosis, in superficiebus planis intergeniculorum positis, 2–4 (vulgo 2) in superficiebus quibusque, singulis aut binis, in seriebus verticalibus per axes dispositis, poris centralibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 545756; Moss Beach, San Mateo county, coast of central California, west coast of North America. Coll. A. V. Manza.

This species has thin basal segments and the primary branching very irregular, but is ultimately regular dichotomous, so that the fronds give the impression of a regular dichotomy.

4. *Bossea frondifera* sp. nov.—B. frondibus 2–7 cm. longis; ramificatione plumoso-pinnatodecomposita oppositaque, ramis ramulisque juvenis ovatis; intergeniculis subbasalibus cylindricis, 1 mm. longis, 1 mm. latis, iis partium superarum cuneatis obcordatisve, 1–2 mm. longis, 1–3 mm. latis, lobis rotundatis; conceptaculis tetrasporangialibus conicis 2 (rare multioribus) in superficiebus planis quibusque prope ad lobos superos singulo positis, poris centralibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 545757; Moss Beach, San Mateo County, coast of central California, west coast of North America. Coll. A. V. Manza.

This species has slender basal segments, branches rather crowded and the young branches and branchlets always ovate. The fronds appear as if

crowded with ovate leaves, hence the designation is chosen to signify the leafy appearance of the fronds.

5. *Bossea Gardneri* sp. nov.—B. frondibus 5–18 cm. longis; ramificatione dichotoma aut primaria laterali sed ultime dichotoma; intergeniculis subbasalibus gracilibus, cylindricis aut lente compressis, 1 mm. longis, 1 mm. latis, iis partium superarum compressis, iis ramorum primariorum cuneatis, 1–2 mm. longis, 1–3 mm. latis, lobis, obtusis, iis ramellorum obcordatis alis tenuibus, lobis rotundatis, et costa prominente indutis, 2–3 mm. longis, 2–4 mm. latis; conceptaculis tetrasporangialibus semiglobosis, 2–6 (vulgo 4) in superficiebus planis intergeniculorum quibusque, 1–3 in alis quibusque indutis, per costam mediam in seriebus longitudinalibus dispositis, poris centralibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 545763; Pacific Grove, Monterey County, coast of central California, west coast of North America. Coll. N. L. Gardner.

This species has more slender basal segments than those of *Bossea Orbigniana*, and likewise it has generally 4 conceptacles on each flat surface of the intergeniculum as compared to 2 in *Bossea Orbigniana*.

6. *Bossea interrupta* sp. nov.—B. frondibus 5–13 cm. longis; ramificatione bi-tri-pinnata oppositaque, ramis in sectoribus ab sectoribus eramosis separatis oriendis; intergeniculis subbasalibus crassis, cylindricis 1–2 mm. longis, 2 mm. latis, iis partium superarum compressis, obcordatis tenui-alatis, obtuse lobatis, iis ramorum 2 mm. longis, 2–4 mm. latis, iis ramulorum spatulatis-3 mm. longis, 1–2 mm. latis; conceptaculis tetrasporangialibus semiglobosis, in quibusque superficiebus planis 2, singulo in lobo quoque prope marginem superam alae quaeque, ad costam propinquissimis, in ramulis singulis centralibusque; poris centralibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 545969; Pacific Grove, Monterey County, coast of central California, west coast of North America. Coll. A. V. Manza.

This species has thick basal segments and compound branches arising in groups separated from one another by simple branchlet-bearing segments.

7. *Bossea Orbigniana* (Decne.) comb. nov.—*Amphiroa* (*Arthrocardia*) *Orbigniana* Decaisne, Mem. sur les Corallines, pp. 112, 1842.

Fronds 7–21 cm. long; branching wholly or partially dichotomous; intergenicula near the base cylindrical, 1–2 mm. long and 2 mm. broad, those of the upper parts compressed cuneate with lobes acute, or obcordate with lobes obtuse, 2 mm. long and 2–3 mm. broad; tetrasporic conceptacles conical on the flat surfaces of the intergenicula, 2 on each surface borne singly on each wing near the upper margins of the lobes very close to the midrib, with pores central.

Species type: *Amphiroa* (*Arth.*) *Orbigniana* Decaisne (loc. cit., p. 112).

Type locality: "Patagonia S. Carlos Chiloensisque littora."

Synonyms: *Amphiroa Orbigniana* Decne., Harvey, Ner. Austr., p. 100, pl. 38, figs. 1–6, 1847; Areschoug, in J. G. Agardh, Sp. Alg., 2, part 2: 539, 1852. *Cheilosporum Orbignianum* (Decne.) Yendo, a revised list of Corallinae, Jour. Coll. Sci. Imp. Univ. Tokyo, Japan, 20, art. 12: 20, 1905.

The type specimen is not available for examination, but in our collection of the Articulated Corallines we have plants which have been referred to this species and which show characters that are consistent not only with the characters of the type, but likewise with those of the species of *Bossea*. It has comparatively thick basal segments and the conceptacles, generally 2 on each flat surface of the intergeniculum, are borne singly on the wings.

8. *Calliarthron latissimum* (Yendo) comb. nov.—*Cheilosporum latissimum* Yendo, Corallinae Japonicae, Jour. Coll. Sci. Imp. Univ. Tokyo, Japan, 16, art. 2:21, pl. 2, figs. 16–17, pl. 6, fig. 7, 1902; a revised list of Corallinae, *Ibid.*, 20, art. 12: 20, 1905.

Fronds 7 cm. long; branching pinnate and opposite to subalternate; intergenicula near the base cylindrical or slightly compressed, 1–2 mm. long and 2 mm. broad, on the upper parts compressed obcordate with lobes rounded, 2–3 mm. long and 3–5 mm. broad; tetrasporic conceptacles both along the margins and on the flat surfaces of the intergenicula.

Species type: *Cheilosporum latissimum* Yendo (loc. cit., p. 21, pl. 2, figs. 16–17, pl. 6, fig. 7, 1902; loc. cit., p. 20, 1905).

Type locality: Prov. of Kazusa, Japan.

Topotype specimen: tetrasporic, Herb. Univ. Calif. No. 418147.

The exact type of this species has not been examined, but a topotype seemingly collected and determined by Yendo himself is available for study and has all the essential characters of the species of *Calliarthron*. It has broader and thicker segments than those of *C. yessoense* (Yendo) comb. nov.

9. *Calliarthron modestum* (Yendo) comb. nov.—*Cheilosporum anceps* var. *modesta* Yendo, Corallinae verae Japonicae, Jour. Coll. Sci. Imp. Univ. Tokyo, Japan, 12, art. 3:19, pl. 2, fig. 9, pl. 6, fig. 3, 1902; a revised list of Corallinae, *Ibid.*, 20, art. 12:20, 1905.

Fronds 5 cm. long; branching pinnate and opposite to subalternate; intergenicula near the base cylindrical or subcompressed, 1 mm. long and 1 mm. broad, and those of the upper parts compressed mostly obcordate with lobes acute, 2 mm. long and 2–3 mm. broad; tetrasporic conceptacles mostly along the lateral margins and flat surfaces of the intergenicula, but occurring sometimes on the apices of simple ramules.

Species type: *Cheilosporum anceps* var. *modesta* Yendo (loc. cit., p. 19, pl. 2, fig. 9, pl. 6, fig. 3, 1902; loc. cit., p. 20, 1905).

Type locality: "Hakodate, Japan."

The exact type of this species is not available, but we have an excellent specimen of the variety, whose designation is in the handwriting of Yendo, and which shows the characters of species of *Calliarthron*. It is the only species that we know of *Calliarthron* where terminal conceptacles have been found.

10. *Calliarthron pinnulatum* sp. nov.—C. frondibus 7–11 cm. longis; ramificatione plumoso-bi-tri-pinnata oppositaque (rare dichotomo-pinnata), divisionibus primariis ramulis simplicibus et ramis longis semel usque ad bis divisis compositis, divisionibus ramulorum ultimorum simplicibus; intergeniculis subbasalibus crassis, cylindricis aut lente compressis, 1–3 mm. longis, 2 mm. latis, partium superarum iis ramorum longorum compressis, cuneatis, 2–3 mm. longis, 2–3 mm. latis, iis ramulorum vulgo spatulatis, rare subulatis, 2–4 mm. longis, 1–2 mm. latis; conceptaculis tetrasporangialibus conicis in superficie plana quaque 2, prope lobos superos ponendis, in superficiebus planis ramulorum singulis centralibusque, poris centralibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 545963; Moss Beach, San Mateo County, coast of central California, west coast of North America. Coll. A. V. Manza.

This is the only species of the genus known thus far with the branching plumosely pinnate and opposite. The basal segments (stipe) are comparatively long and thick.

11. *Calliarthron regenerans* sp. nov.—C. frondibus 7–15 cm. longis, ramificatione interruptopinnata oppositaque aut dichotomo-interruptopinnata oppositaque; intergeniculis subbasalibus cylindricis aut lente compressis, 1–4 mm. longis, 1–2 mm. latis, iis partium superarum omnino compresso-obcordatis, rotundato-lobatis, aut in ramis primariis, compressis aut in ramulis, cylindricis 2–3 mm. longis, 1–4 mm. latis; conceptaculis et in marginibus lateralibus et in superficiebus planis intergeniculorum positis, in marginibus 2–3 et in superficiebus planis congestis, semiglobosis in plantis tetrasporangiferis, conicis apice obtusis in plantis cystocarpiferis, et conicis apice acuminatis in plantis antheridiiferis.

Species types: tetrasporic, Herb. Univ. Calif. No. 545737; cystocarpic, Herb. Univ. Calif. No. 545738; antheridial, Herb. Univ. Calif. No. 545775; Moss Beach, San Mateo County, coast of central California, west coast of North America. Coll. A. V. Manza.

This species has comparatively long and thick basal segments. The branching is generally interruptedly pinnate and opposite. The branches arise in groups, each separated by branchless segments. The segments of the branches may be entirely compressed, or the lower portions of the branches may be compressed while their apices may be cylindrical.

12. *Calliarthron Schmittii* sp. nov.—C. frondibus verisimiliter repentibus et dorsiventralibus; ramificatione dichotoma aut dichotomo-laterali; intergeniculis subbasalibus cylindricis, 1–4 mm. longis, 1 mm. latis, iis partium superarum compressis, convexis, valde costatis, suborbicularibus subcordatisve, marginibus integris aut irregulariter lobatis, 5–10 mm. longis, 5–15 mm. latis; conceptaculis tetrasporangialibus semiglobosis, in superficiebus superis intergeniculorum restrictis, per costam dispositis; poris centralibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 545744; dredged from 21–24 fathoms, Point Loma, coast of southern California, west coast of North America. Coll. W. Schmitt, D-4303 (U. S. F. C. Str. *Albatross*, 1904).

Of this species we have merely fragmentary specimens. All indications, however, seem to point out that the fronds are creeping. This is the only species of the genus thus far known with the intergenicula convex, dorsiventral and the conceptacles restricted to the convex (upper) surfaces.

13. *Calliarthron Setchelliae* sp. nov.—C. frondibus 9–16 cm. longis; ramificatione dichotomopinnata oppositaque; intergeniculis subbasalibus vulgo compressis, 1–3 mm. longis, 2–3 mm. latis, iis partium superarum compresso-cuneatis obcordatisve, 2–3 mm. longis, 3–5 mm. latis, lobis obtusis; conceptaculis tetrasporiiferis semiglobosis, cystocarpiiferis apice obtusis, et antheridiiferis conicis apice acuminatis, omnibus per margines laterales intergeniculorum oriendis, 2–4 in margine quaque, aetate provecta per superficies planas intergeniculorum sparsis.

Species types: tetrasporic, Herb. Univ. Calif. No. 545733; cystocarpic, Herb. Univ. Calif. No. 548906; antheridial, Herb. Univ. Calif. No. 548904; Moss Beach, San Mateo County, coast of central California, west coast of North America. Coll. A. V. Manza.

The segments of this species are practically all compressed; the branching on the lower parts is very irregular, but ultimately it tends to be dichotomous; the stipe is comparatively very short.

14. *Calliarthron yessoense* (Yendo) comb. nov.—*Cheilosporum yessoense* Yendo, Corallinae verae Japonicae, Jour. Coll. Sci. Imp. Univ. Tokyo, Japan, 16, art. 3:19, pl. 2, figs. 12–13, pl. 6, fig. 5, 1902; a revised list of Corallinae, *Ibid.*, 20, art. 12:30, 1905.

Fronds 7 cm. long; branching pinnate and subalternate; intergenicula near the base cylindrical, 1–2 mm. long and 1 mm. broad, those of the upper parts compressed, obcordate with lobes rounded, 2 mm. long and 2–6 mm. broad; tetrasporic conceptacles along the lateral margins and on the flat surfaces of the intergenicula.

Species type: *Cheilosporum yessoense* Yendo (loc. cit., p. 19, pl. 2, figs 12–13, pl. 6, fig. 5, 1902; loc. cit., p. 20, 1905).

Type locality: "Hakodate, Japan."

Topotype specimen: tetrasporic, Herb. Univ. Calif. No. 90783.

The type of this species is not available, but we have a topotype specimen seemingly collected and determined by Yendo himself, which shows the characters of *Calliarthron*. The segments of this species are much thinner than those of *C. latissimum* (Yendo). Manza.

15. *Joculator maximus* (Yendo) comb. nov.—*Cheilosporum maximum* Yendo, Corallinae verae Japonicae, Jour. Coll. Sci. Imp. Univ. Tokyo, Japan, 16, art. 3:22, pl. 2, figs. 18–19, pl. 6, fig. 9, 1902; a revised list of Corallinae, *Ibid.*, 20, art. 12:26, 1905.

Fronds 12 cm. long; branching plumosely bi-tripinate and opposite, with the divisions mostly of simple ramules on the lower parts and of long branches on the upper parts, and entirely of simple ramules on the branchlets; intergenicula near the base cylindrical, 1–2 mm. long and 1–1.5 mm. broad, those of the upper parts on the branches compressed cuneate, 1–2 mm. long and 2–3 mm. broad, and those of the ramules spathulate, 2–4 mm. long and 1–2 mm. broad; tetrasporic conceptacles both terminal on the apices of the ramules with pores apical, and lateral on the flat surfaces of the intergenicula, 2 on each surface borne singly near the upper lobes, with pores central.

Species type: *Cheilosporum maximum* Yendo (loc. cit., p. 22, pl. 2, figs. 18–19, pl. 6, fig. 9, 1902; loc. cit., p. 26, 1905.

The exact type locality is most uncertain. Yendo (loc. cit., p. 22, 1902) cited only the general distribution (Prov. of Boshu; Prov. of Kazusa; Misaki; Shimoda).

The type of this species is not available for examination, but specimens seemingly collected and determined by Yendo himself have been studied. The plants have all the characters of *Joculator*. The species is much larger than *Joculator pinnatifolius* Manza.

NEW SPECIES OF ARTICULATED CORALLINES FROM SOUTH AFRICA

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There have been received from Professor T. A. Stephenson, from Dr. G. F. Papenfuss and from Dr. A. V. Duthie, extensive collections of the Articulated Corallines of the coasts of South Africa for examination. A study of these collections has shown that certain of these species must be described as new. The proposed names, descriptions and other details are outlined below, together with one new combination. It may be noted that the specimens received from Professor Stephenson were collected by the "Ecological Survey of the University of Cape Town" and will, therefore, be designated "Ecol. Surv." in the following account.

1. *Arthrocardia attenuata* sp. nov.—A. frondibus 4–6 cm. longis; ramificatione vegetativa bitripinnata, ramis subalternis, fructifera cymoidea; intergeniculis subbasalibus cylindricis, 1–2 mm. latis, iis partium superarum ramorum primariorum compresso-cuneatis aut obcordatis, lobis acutis, 2 mm. longis et 2 mm. latis, iis ramulorum ultimorum gracilibus filiformibusque; conceptaculis tetrasporangialibus terminalibus, cymoideo-fasciculatis, poris apicalibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 548786; Sea Point, South Africa. Coll. Dr. A. V. Duthie.

Arthrocardia attenuata differs from all the species of *Arthrocardia* thus far known by its ultimate segments becoming slender and filiform.

2. *Arthrocardia Gardneri* sp. nov.—A. frondibus 4–6 cm. longis; ramificatione vegetativa bi-tripinnata, ramis oppositis, longis, semel aut bis divisis, fructifera cymoidea; segmentis subbasalibus crassis, deorsum attenuatis, intergeniculis cylindricis, 1–2 mm. longis, 1–2 mm. latis, iis partium superarum plerumque compressis, intergeniculis ramorum primariorum, obovatis, obcordatis aut spatulatis, 2–4 mm. longis, 2–3 mm. latis et iis ramulorum ultime gracilibus; conceptaculis tetrasporangialibus terminalibus, cymoideo-fasciculatis, poris apicalibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 565485; Isipingo Beach, Natal, South Africa. Coll. Ecol. Surv. D.C.11.

The branching of the vegetative parts in this species is regularly pinnate and opposite as compared to pinnate and subalternate in *Arthrocardia attenuata*. Although the branchlets in *A. Gardneri* likewise become slender, they remain compressed, while in *A. attenuata* the slender branchlets become cylindrical.

3. *Arthrocardia linearis* sp. nov.—A. frondibus 3–8 cm. longis; ramificatione vegetativa pinnato-decomposita, ramis oppositis, longis, semel usque ad bis pinnato-divisis, divisionibus ultimis ramulis gracilibus, simplicibus aut paulo divisis, fructifera cymoidea; segmentis subbasalibus cylindricis intergeniculis 1–3 mm. longis, 1–1.5 mm. latis; iis partium superarum compressis; intergeniculis ramorum primariorum obcordatis, lobis obtusis acutisve, 2–3 mm. longis, 2–3 mm. latis; et iis ramulorum plerumque linearibus, 3–6 mm. longis, 1 mm. latis, aliquando filiformibus; conceptaculis tetrasporangialibus terminalibus, cymoideo-fasciculatis, poris apicalibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 564560; Isipingo Beach, Natal, South Africa. Coll. Ecol. Surv. D.C. 3.

The fronds of *Arthrocardia linearis* are plumosely pinnate, with rather thick basal segments. The divisions of the branches consist of long branches and of ramules. The ramules are elongated spathulate, but mostly linear.

4. *Arthrocardia Papenfussii* sp. nov.—A. frondibus 3–5 cm. longis; ramificatione vegetativa pinnato-decomposita et opposita, ramis longis semel usque ad bis pinnato-divisis, divisionibus ultimis ramulis simplicibus compositis, fructifera cymoidea; segmentis subbasalibus deorsum attenuatis, intergeniculis cylindricis, 2–3 mm. longis, 1–1.5 mm. latis; iis partium superarum ramorum primariorum compressis, leviter subcompresso-cuneatis, 2 mm. longis, 2–3 mm. latis, et iis ramulorum leviter compressis, subclavatis, 3 mm. longis, 1–1.5 mm. latis; conceptaculis terminalibus, cymoideo-fasciculatis, poris apicalibus.

Species types: tetrasporic, Herb. Univ. Calif. No. 564568; Coll. G. F. Papenfuss No. 50; cystocarpic, Herb. Univ. Calif. No. 564583; Coll. G. F. Papenfuss No. 50-A; Melkbosch, South Africa.

Arthrocardia Papenfussii has rather thick segments, slightly compressed to almost cylindrical in the tetrasporic plants. In the cystocarpic plants which are smaller than the tetrasporic, the segments are more compressed and the branches arise almost at right angles from the axes producing them.

5. *Arthrocardia Setchelliae* sp. nov.—A. frondibus 3–4 cm. longis; ramificatione vegetativa plumoso-pinnata-tripinnata, opposita, ramis primariis elongatis, semel usque ad bis pinnato-divisis, divisionibus ultimis ramulis simplicibus compositis, fructifera cymoidea; segmentis partium inferorum deorsum attenuatis usque ad filiformibus gracilibusque; iis partium superarum compressis; iis ramorum primariorum crasse cuneatis obovatisve, 2 mm. longis, 1 mm. latis; conceptaculis tetrasporangialibus terminalibus, maxime cymoideo-fasciculatis, aliquando apices ramulorum simplicium terminantibus; poris apicalibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 564557; Witsands, South Africa. Coll. G. F. Papenfuss No. 57.

Arthrocardia Setchelliae is a small feather-like species in which the axial segments of the branches and of the branchlets are comparatively thick. The basal segments are cylindrical and filiform.

6. *Arthrocardia Setchellii* sp. nov.—A. frondibus 3–5 cm. longis; ramificatione vegetativa pinnata oppositaque aut pinnato-decomposita, ramis longis semel usque ad bis divisis et ramis curtis ramulis simplicibus compositis; segmentis subbasalibus, leviter deorsum attenuatis, intergeniculis leviter compressis aut cylindricis, 1 mm. longis, 2–3 mm. latis; iis ramorum superorum compressis, obcordatis, lobis maxime obtusis, rare acutis, 2 mm. longis, 2–3 mm. latis; conceptaculis tetrasporangialibus terminalibus cymoideofasciculatis, sed per occasionem apices ramulorum simplicium terminantibus, poris apicalibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 564553; Port Nolloth, South Africa; coll. Ecol. Surv. P.N.C. 5.

Arthrocardia Setchellii has segments much thicker than those of *A. Stephensonii*, and the basal segments are comparatively simple and thick.

7. *Arthrocardia Stephensonii* sp. nov.—A. frondibus 2–4 cm. longis; ramificatione vegetativa pinnata oppositaque aut pinnato-decomposita, fructifera cymoidea; segmentis subbasalibus gracilibus filiformibusque, simplicibus aut in ramulos graciles proliferentibus; iis partium superarum compressis obcordatisque lobis maxime acutis rare obtusis, 1–2 mm. longis 1–2 mm. latis; conceptaculis tetrasporangialibus terminalibus, cymoideofasciculatis, poris apicalibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 564581; St. James, Cape Peninsula, South Africa; coll. Ecol. Surv. F-121.

Arthrocardia Stephensonii is a comparatively small species, even smaller than *A. Setchellii*. The basal segments in *A. Stephensonii* are slender, filiform, proliferous into slender branchlets and with the branchlets seemingly intertwining.

8. *Cheilosporum africanum* sp. nov.—C. frondibus 5–8 cm. longis; ramificatione primaria constanter dichotoma, secundaria laterali, varie e intergeniculis orienda, dichotome divisa aut subpinnata, apicibus ramulorum ramulis digitatis dense instructis; segmentis subbasalibus leviter deorsum attenuatis, intergeniculis inferis cylindricis, 1 mm. longis, 1–1.5 mm. latis; iis partium superarum compressis, ultimis gracilibus; iis ramorum vestustiorum intergenicula ramorum primariorum obcordata, lobis acutis obtusisve, iis ramulorum obcordata, lobis acuminatis, 1–2 mm. longa, 1–3 mm. lata ostendentibus; conceptaculis tetrasporangialibus singulo in marginibus superis loborum superorum proxime ad axillas intergeniculorum positis, poris apicali-marginalibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 564605; Still Bay, South Africa; Coll. Ecol. Surv. S.B.-80.

The intergenicula of the branches of this species are regularly obcordate with the lobes of the intergenicula, acute to acuminate as contrasted to the cuneate or obcordate intergenicula with the apices of the upper lobes obtuse or acute in *Cheilosporum flabellatum* (Harvey) Aresch. The segments of the apices of the older branches in *C. flabellatum* are slender and seemingly cylindrical, while in *C. africanum*, although they are likewise slender, they are compressed. In addition, in *C. flabellatum*, the branching, according to all indications, is dichotomous while in *C. africanum*, short lateral proliferations arise from the dichotomies in the older parts, a condition not as yet observed in *C. flabellatum*.

9. *Cheilosporum multifidum* (Kuetz.) comb. nov.—*Amphiroa multifida* Kuetzing, Tab. Phyc., 8:27, tab. 56, fig. 1, 1858.

Fronds 3–5 cm. long; branching largely dichotomous but sometimes with lateral proliferations; segments of the lowermost parts slender filiform and proliferous; those of the upper parts compressed with the intergenicula obcordate or somewhat obcordate, with lobes of the intergenicula of the lower parts of the branches obtuse, acute or cleft and of those near the apices fimbriate, 1 mm. long and 0.204 mm. broad; tetrasporic conceptacles on the upper margins of the lobes of the intergenicula, borne singly near the axils, with pores apical-marginal.

Species type: *Amphiroa multifida* Kuetzing (loc. cit., p. 27, tab. 56, fig. 1).

Type locality: Cape of Good Hope.

Synonyms: *Amphiroa multifida* Kuetzing (loc. cit., p. 27, tab. 56, fig. 1). *Cheilosporum cultratum* f. *multifida* Yendo, a revised list of Corallinae, Jour. Coll. Sci. Imp. Univ. Tokyo, Japan, 20, art. 12: 19, 1905.

The type specimen of the species is not available but Kuetzing in his account gives excellent illustrations of the type plant showing characters essentially similar to our plants. This species, according to our generic limitation, belongs to *Cheilosporum* and not to *Amphiroa*. It differs from *C. cultratum* by the lobes of the intergenicula being fimbriate, while in the latter the lobes are entire.

10. *Jania digitata* sp. nov.—J. frondibus 5–8 cm. longis; ramificatione dichotomo-digitato-cymoidea; intergeniculis subbasalibus cylindricis, 1–2 mm. longis, 1 mm. latis; iis partium superarum ramorum primariorum cylindricis, lente compressis, clavatis subclavatisve et iis ramorum ultimorum gradatim gracilibus, cylindricis aut leviter compressis, apicibus extremis lente tumidis, 2–3 mm. longis, 1–1.5 mm. latis; conceptaculis tetrasporangialibus terminalibus, cymoideo-fasciculatis, ultimis antennas duas ferentibus, poris apicalibus.

Species type: tetrasporic, Herb. Univ. Calif. No. 564574; St. James, Cape Peninsula, South Africa; Coll. Ecol. Surv. F.-123.

Jania digitata is the largest species of the genus yet known. The general branching is dichotomous-cymoid, but oftentimes three or more branches arise digitately, in one plane, from the apices of the intergenicula.

THE STABILITY OF THE RADIATIVE GRADIENTS IN THE INTERIOR OF A STAR

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If a spherically symmetrical distribution of matter is in hydrostatic equilibrium and if, further, radiative equilibrium obtains, then the radiative temperature gradient is determined by [see Note 1 at the end of the paper]

$$\frac{dT}{T} = \frac{1}{4}\,\frac{\kappa\eta}{\overline{\kappa\eta}(r)}\,\frac{dP}{P}, \tag{1}$$

where κ is the opacity coefficient,

$$\eta = \frac{L(r)}{M(r)}\Big/\frac{L}{M}; \quad \overline{\kappa\eta}(r) = \frac{1}{P}\int_0^P \kappa\eta\, dP, \tag{2}$$

$L(r)$ the amount of radiant energy crossing a spherical surface of radius r, and $M(r)$ is the mass interior to r. Finally

$$P = \frac{k}{\mu H}\,\rho T + \frac{1}{3}\,aT^4 \tag{3}$$

where k, a, μ and H are respectively the Boltzmann constant, the Stefan-Boltzmann (radiation) constant, the molecular weight and the mass of the proton.

To examine the stability of the radiative gradient we suppose that an element of mass δm originally at temperature T, density ρ and pressure P suffers a sudden increase of temperature of amount $\delta T > 0$. Then this element would exert a pressure of a definite amount $\delta P > 0$ on its surroundings thus expanding and becoming less dense than its immediate neighborhood. The element δm would then experience a force tending to displace it into regions of lower density. During such a movement the element would continue to expand, the temperature altering meantime.

We shall now make the following assumptions: (1) at each instant of time the element δm expands (or contracts) to such an extent that the pressure exerted on the element by the surrounding material is the same as the element exerts on the surrounding material; (2) that the process of expansion (or contraction) takes place adiabatically; (3) that the viscous forces restricting the movement of the element δm can be neglected. We shall first examine the consequences of these assumptions and later subject them to a closer scrutiny.

By our second assumption, since the expansion of δm takes place adiabatically, we should have [see Note 2 at the end of the paper]:

$$\frac{dT}{T} = \frac{\Gamma_2 - 1}{\Gamma_2} \frac{dP}{P}, \tag{4}$$

where

$$\Gamma_2 = 1 + \frac{(4 - 3\beta)(\gamma - 1)}{\beta^2 + 3(\gamma - 1)(1 - \beta)(4 + \beta)}, \tag{5}$$

$$\beta P = \frac{k}{\mu H} \rho T; \quad (1 - \beta)P = \frac{1}{3} aT^4. \tag{6}$$

(4) governs the rate of change of the temperature of the element δm as it moves outward into regions of lower density, expanding meantime.

Comparing (1) and (4) we see that the temperature of δm as it moves outward alters at a rate different from that of its immediate surroundings, because according to our first assumption P alters in the same way for both δm and the surroundings.

Let us now suppose that

$$\frac{\Gamma_2 - 1}{\Gamma_2} > \frac{1}{4} \frac{\kappa\eta}{\overline{\kappa\eta}(\gamma)}. \tag{7}$$

Then it follows that the element δm after moving outward for a certain distance will find itself at the same temperature, pressure and density as its surroundings at that point and consequently the original disturbance eventually dies out.

In the same way, if the element δm originally suffers a *decrease* of temperature of amount δT then it would become denser than its immediate surroundings and would consequently sink to regions of higher density. If we make our three assumptions as before, then the adiabatic compression it experiences as it sinks to regions of higher density increases the temperature of δm at a rate greater than does its surroundings. Again it will soon find itself at a point where δm and its neighborhood at that point have the same pressure, density and temperature. Thus in either case—i.e., either a positive or a negative increment δT of the element δm—the dis-

turbance dies out if (7) is true. In this sense, the radiative equilibrium is stable if the radiative gradient is less than the corresponding adiabatic gradient—a result first proved by Schwarzschild, but the assumptions on which it can be proved are generally not made explicit.

On the other hand if

$$\frac{\Gamma_2 - 1}{\Gamma_2} < \frac{1}{4}\frac{\kappa\eta}{\overline{\kappa\eta}(r)} \tag{8}$$

then if the element δm suffers an increase (decrease) of temperature, then as before it would move outward (downward) to regions of lower (higher) density but now the temperature of δm decreases (increases) less than does that of its surroundings, and hence is always at a temperature higher (lower) than its surroundings. Hence we have proved on the basis of our assumptions that *the radiative equilibrium is stable or unstable according as*

$$4\frac{\Gamma_2 - 1}{\Gamma_2} > \text{or} < \frac{\kappa\eta}{\overline{\kappa\eta}(r)}. \tag{9}$$

The following table gives the values of $4(\Gamma_2 - 1)/\Gamma_2$ for different values of $(1 - \beta)$:

$1-\beta$	$4(\Gamma_2-1)/\Gamma_2$	$1-\beta$	$4(\Gamma_2-1)/\Gamma_2$	$1-\beta$	$4(\Gamma_2-1)/\Gamma_2$
0	1.6	0.4	1.065	0.8	1.004
0.1	1.304	0.5	1.039	0.9	1.000
0.2	1.177	0.6	1.022	1.0	1.0000
0.3	1.108	0.7	1.010	...	.....

Suppose now that we have initially a situation in which the radiative gradient obtains but one which exceeds the adiabatic gradient. By our above discussion the radiative gradient is unstable in the sense that a slight alteration of the local temperature will give rise to a system of ascending and descending currents which would have the effect of reducing the existing radiative temperature gradient. It is clear that eventually a steady state would be set up though it is not *a priori* clear what the nature of that steady state would be. We shall continue our discussion on the basis of the three fundamental assumptions we have already made. It is generally assumed that eventually the temperature gradient that would be set up would very approximately be the adiabatic one. Actually, of course, the adiabatic gradient would not be rigorously set up. For there will always be a certain amount of transport of heat energy by radiation and the presumption is that eventually there would be a "slight" *excess* of the currents in the direction of increasing temperature such that the net transport of heat energy due to the currents is positive in the direction of increasing temperature and is further of an amount *exactly equal* to the transport of heat energy by the radiation in the opposite direction. Now the

transport of heat energy by the radiation will be governed by the equation

$$dE_r = cdp_r = -\frac{\kappa L(r)}{4\pi r^2}\rho dr. \tag{10}$$

If this were so, it follows that the law of the conservation of momentum would require the excess currents in the direction of increasing temperature to transfer momentum of an amount exactly that transferred by radiation in the opposite direction. Hence on these premises the equations of equilibrium are clearly

$$\frac{k}{\mu H}\frac{d}{dr}(\rho T) = -\frac{GM(r)}{r^2}\rho, \tag{11}$$

$$\frac{dP}{P} = \frac{\Gamma_2}{\Gamma_2 - 1}\frac{dT}{T}. \tag{12}$$

It should be remarked that in the literature[3] equation (12) for the temperature gradient has been mistakenly combined with

$$\frac{d}{dr}\left(\frac{k}{\mu H}\rho T + \frac{1}{3}aT^4\right) = -\frac{GM(r)}{r^2}\rho. \tag{13}$$

Equations (12) *and* (13) *together form an inconsistent set of equations, for the use of* (13) *as the equation of hydrostatic equilibrium already implies the existence of the radiative temperature gradient.* The simplest way of seeing this is perhaps the following: Equation (13) assumes that $dp_r(\nu)d\nu$ represents the rate of transfer of momentum across an elementary infinitesimal cylindrical slab at r due to radiation in the frequency interval $(\nu, \nu + d\nu)$. On the other hand it is a general theorem that the mechanical force exerted by radiation is given by $-\kappa_\nu \rho F_r dr/c$. These two statements together imply precisely the existence of the radiative temperature gradient which contradicts the use of (12) in combination with (13).

We have now to examine the fundamental assumptions on which our discussion has so far been carried. The first important point to realize is that if the radiation pressure is important—in other words if β and $(1 - \beta)$ are of comparable orders of magnitude—then the system of "convection" currents set up must be quite "violent." For, as we have seen, the excess currents in the direction of increasing temperature transfer momentum of amount exactly equal to that transferred by radiation in the opposite direction, and hence the resulting pressure due to mass motion must be of the same order as the gas kinetic pressure, and this implies a state of quite active stirring. It is then *a priori* probable that the *inertia of motions* is still quite large, in which case the elements of ascending and descending masses would experience considerable viscous friction (contrary to one of our hypotheses that the viscous friction could be neglected) which in turn

would communicate to the elements appreciable amounts of heat contradicting our assumption (2) that the expansion (or contraction) of the elements take place adiabatically. Finally our hypothesis (1) that at each instant of time the element δm expands (or contracts) to such an extent that the pressure exerted on the element δm by the surrounding material is the same as the element exerts on the surrounding material, assumes that during the movements of the element of mass the equalization of density in the element due to the diffusion of the surrounding material is negligible, which need not in general be the case.

On the whole, it must be concluded that we cannot at present make any definite predictions about the nature of the "convection currents" that would be set up if the radiation pressure is important and if further the radiative gradient exceeds the adiabatic. It, however, seems probable that in the limit of vanishing radiation pressure the equations (11) and (12) may approximate very closely to the actual equations of equilibrium.

In conclusion, I wish here to record my appreciation of the valuable discussions I have had on the subject of this note with Professors B. Strömgren and G. P. Kuiper.

NOTE 1. The equations of equilibrium for material in radiative equilibrium are

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho, \tag{i}$$

$$\frac{dp_r}{dr} = -\frac{\kappa L(r)}{4\pi cr^2}\rho, \tag{ii}$$

$$P = p_r + p_g = \frac{k}{\mu H}\rho T + \frac{1}{3}aT^4. \tag{iii}$$

Dividing (ii) by (i) and remembering our definition of η we have

$$dp_r = \frac{L}{4\pi cGM}\kappa\eta dP, \tag{iv}$$

or integrating

$$p_r = \frac{L}{4\pi cGM}\int_0^P \kappa\eta dP, \tag{v}$$

which can also be written as

$$p_r = \frac{L}{4\pi cGM}\bar{\eta\kappa}(r)\,.\,P. \tag{vi}$$

By (iv) and (vi) we have

$$\frac{dp}{p_r} = \frac{\kappa\eta}{\bar{\kappa\eta}(r)}\frac{dP}{P}, \tag{vii}$$

or

$$\frac{dT}{T} = \frac{1}{4}\frac{\kappa\eta}{\overline{\kappa\eta}(r)}\frac{dP}{P}, \qquad \text{(viii)}$$

which is the equation quoted.

NOTE 2. For an adiabatic change we should have

$$\left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT + PdV = 0, \qquad \text{(i)}$$

where U is the internal energy of the system. For our case we have

$$U = aT^4V + c_VT; \quad P = (c_p - c_V)\frac{T}{V} + \frac{1}{3}aT^4, \qquad \text{(ii)}$$

where c_p and c_V are the two specific heats of the gas.

From (i) and (ii) we derive that

$$\left(12p_r + \frac{1}{\gamma-1}p_g\right)\frac{dT}{T} + (4p_r + p_g)\frac{dV}{V} = 0. \qquad \text{(iii)}$$

On the other hand Γ_2 is *defined* by the relation

$$\frac{dP}{P} + \frac{\Gamma_2}{1-\Gamma_2}\frac{dT}{T} = 0 \qquad \text{(iv)}$$

or according to (ii)

$$\left[4p_r + p_g + \frac{\Gamma_2}{1-\Gamma_2}(p_r + p_g)\right]\frac{dT}{T} - p_g\frac{dV}{V} = 0. \qquad \text{(v)}$$

Comparing (iii) and (v) we derive the expression for Γ_2 quoted. A condition equivalent to (9) has been given by H. Siedentopf (*A. N.*, **244**, 278 (1932)).

NOTE 3. Bierman, L., *Zeits. f. Ap.*, **5**, 128 (1932); *A. N.*, **257**, 270 (1935).

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*PLANT PARTS OF DECIDUOUS FRUITS WHICH GIVE EVIDENCE OF BEING FAVORABLE FOR TISSUE CULTURE**

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White,[6,7] in an early review of plant tissue cultures, called attention to the fact that, as in animal tissue cultures, certain plant tissues might be found to respond to treatment more readily than others. Among these he suggested young embryos, root tips and seed primordia. Root tips have since proved well suited to culture and manipulation *in vitro*, as called to attention by Robbins,[2] and as developed by White,[8] Robbins and Bartley[3] and others. More recently Bonner[1] has shown that the

inner surface of the bean pod responds to treatment with various tissue extracts. The purpose of this paper is to record three plant parts of the peach (*Prunus persica* Batsch.) which, in course of culture of embryos of deciduous fruits,[5] appeared favorable to development in culture. These tissues have the further advantage of being easily excised aseptically without recourse to disinfectants.

1. *Micropylar Region of the Integuments.*—At the time of full bloom, when the ovary measured 3.5 mm. in length and nucellus and integuments 0.8 mm. in length, seed excised from developing fruit were placed in White's media[8] on microculture slides. At this stage in development the embryo was still microscopic and the nucellus and integuments were in the stage of rapid development.[4] At maturity the seed of the peach is 20.5 mm. in length.

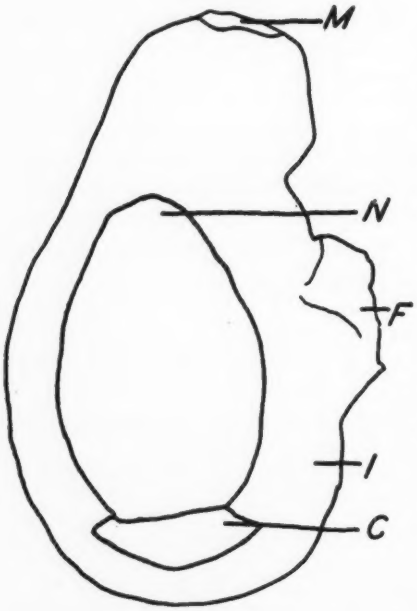


FIGURE 1

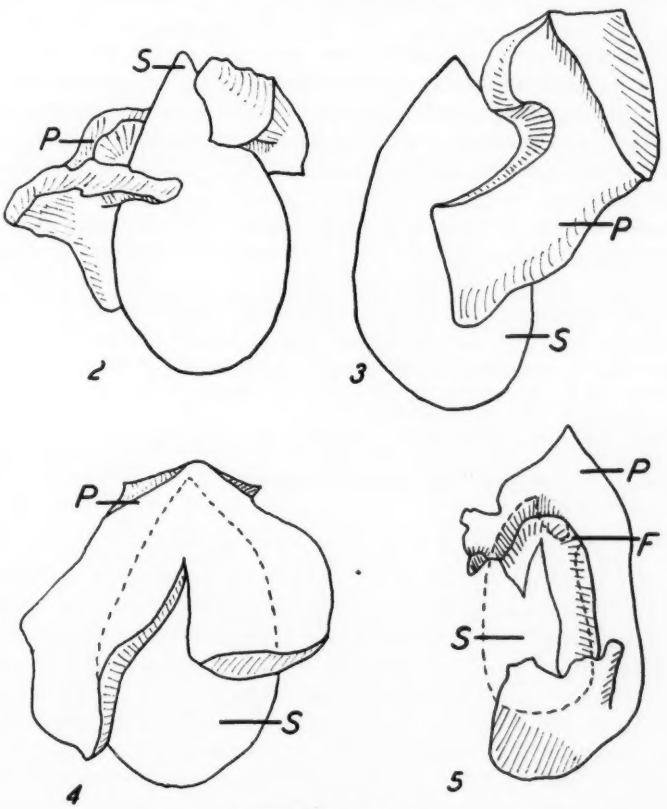
Outline of development of immature peach seed in culture, traced on ground glass by transmitted light, showing enlargement of micropylar region. Increase in length from 0.8 to 1.25 mm. *M*—micropyle; *N*—nucellus; *F*—funiculus; *I*—integuments; *C*—chalaza.

Under the cultural conditions provided, the integuments in the region of the micropyle began immediately to enlarge. Within 5 days the nucellus and integuments had increased from 0.8 to 1.25 mm. in length, the increment being almost entirely in the integuments in the micropylar region (Fig. 1). A gross examination of the tissues involved indicated that the cells in this region had increased in number and that they had enlarged in all their dimensions.

2. *Entire Integuments.*—At 41 days after full bloom, when the ovary was 40.5 mm. in length, the nucellus and integuments 17.1 mm. and the embryo 3.8 mm., seed excised from developing fruits were cultured on agar after treatment with calcium hypochlorite solution.[5] At this stage the integuments are fleshy and about 0.6 mm. thick. In culture they thickened to 1.3 mm. and became brown and relatively tough. Examination showed that the increase in size was

due to the formation of layers of new cells in the outer portion of the outer integument.

3. *Placental Region of the Inner Wall of the Carpel.*—The most favorable responses were secured from the placental region of the inner wall of the



FIGURES 2, 3 AND 4

Development of the placental region of the inner wall of the carpel of the peach in culture, growing nearly around the seed in one instance.

FIGURE 5

Development of an isolated portion of carpel tissue from which the seed had abscissed. Seed 7.5 mm. in length, carpel tissue originally 3 to 10 square millimeters. *S*—seed; *P*—placental region of inner wall of carpel; *F*—funiculus.

carpel, using the varieties Carman and Hiley, 20 and 34 days after full bloom. At 20 days after full bloom Carman fruits were 13.8 mm. in

length, nucellus and integuments were 6.2 and embryos were microscopic. Entire seed, together with 3 to 10 square millimeters of the attached inner wall of the carpel, were removed under aseptic conditions and placed on nutrient agar. In 12 instances the carpel tissue proliferated to many times the original size as shown in figures 2, 3 and 4. In one instance the seed abscissed from the carpel tissue, yet the latter continued to grow (Fig. 5).

Material behaved similarly 34 days after full bloom when the fruits were 28.9 mm. in length, nucellus and integuments 13.6 and embryos 0.12 mm. In 11 days the carpel tissue had increased many times in size. No transfers to new media were made, and no further increase in size was observed 15 days from the initiation of the culture, yet differentiation of the tissue continued, so that in 68 days sclerenchymatous and stony zones appeared, similar to the stony pericarp of the peach into which the tissue would have developed had it not been removed from the fruit for culture.

* Journal Article No. 219 of the New York State Agricultural Experiment Station.

[1] Bonner, James, *Proc. Nat. Acad. Sci.*, **22**, 426–430 (1936).
[2] Robbins, W. J., *Bot. Gaz.*, **73**, 376–390 (1922).
[3] Robbins, W. J., and Bartley, Mary A., *Science*, **85**, 246–247 (1937).
[4] Tukey, H. B., *Proc. Amer. Soc. Hort. Sci.*, **30**, 209–218 (1933).
[5] Tukey, H. B., *Ibid.*, **32**, 313–322 (1935).
[6] White, Philip R., *Archiv für exp. Zellf.*, **10**, 501–518 (1931).
[7] White, Philip R., *Ibid.*, **12**, 601–620 (1932).
[8] White, Philip R., *Plant Phys.*, **9**, 585–600 (1934).

TRANSFINITE REAL NUMBERS[1]

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Let L be a given linear order consisting of abstract elements or symbols β. We define a system F whose elements are certain forms written in terms of the elements of L. We then linearly order these forms, and define *addition* and *multiplication* for them. For convenience of statement, we first define and order certain special elements of F which we term *power-forms* and *product-forms*.

Power-Forms.—We begin by taking all elements β of L as elements of F. Next, all forms β^b, where β belongs to L and b is any real number, are taken to be elements of F. We call these forms *power-forms*. The form β^1 is considered as identical with β; and all forms β^0, no matter what β may be, as identical with 1 (unity).

Product-Forms.—Next, we take as elements of F all forms

$$\pi = \prod_{\lambda<\mu} \beta_\lambda^{b_\lambda} = \beta_1^{b_1}\beta_2^{b_2} \ldots \beta_\omega^{b_\omega} \ldots \beta_\lambda^{b_\lambda} \ldots, \qquad \lambda < \mu,$$

where μ is a finite or transfinite ordinal (variable with π), the exponents b_λ are real numbers and the base elements β_λ are elements of the linear order L satisfying the inequalities:

$$\beta_1 > \beta_2 > \ldots; > \beta_\omega > \ldots > \beta_\lambda > \ldots, \qquad \lambda < \mu.$$

We call such a form π a *product-form* of F.

Product of Two Product-Forms.—We define order for product-forms in terms of *multiplication* for such forms. By the *product* of two power-forms β^{b_1}, and β^{b_2} of F we understand the power-form $\beta^{b_1+b_2}$; of two power forms $\beta_1^{b_1}$ and $\beta_2^{b_2}$, where $\beta_1 > \beta_2$ as the product-form $\beta_1^{b_1}\beta_2^{b_2}$. In general the product of two product-forms is defined as the symbolic algebraic product, exponents of the same base element being summed. This algebraic product may be written as a product-form of F. The *reciprocal*, or inverse with respect to multiplication, of a product-form $\pi = \prod_{\lambda<\mu} \beta_\lambda^{b_\lambda}$, is the product-form $\pi^{-1} = \prod_{\lambda<\mu} \beta_\lambda^{-b_\lambda}$. It may be shown that the product-forms of F constitute an abelian group under multiplication.

Order of Two Product-Forms.—A power-form β^b, with $b \neq 0$, is said to *precede* or *follow* 1 according as b is negative or positive. A product-form $\prod_{\lambda<\mu} \beta_\lambda^{b_\lambda}$, with $b_1 \neq 0$, is said to precede or follow 1 according as its first "factor" $\beta_1^{b_1}$ precedes or follows 1. In general, a product-form π_1 is said to precede a product-form π_2 if the *quotient* $\pi_2/\pi_1 = \pi_2\pi_1^{-1}$ follows 1. It is readily proved that the product-forms of F are thus linearly ordered.

Sum-Form.—We are now ready to define the general element of F, of which power-forms and product-forms are specializations. This general element, which we call a *sum-form* of F, is, namely, any form

$$\sigma = \sum_{\mu<\alpha} a_\mu \pi_\mu$$

$$= \sum_{\mu<\alpha} a_\mu \prod_{\lambda<\lambda_\mu} \beta_{\mu\lambda}^{b_{\mu\lambda}}$$

$$= \sum_{\mu<\alpha} a_\mu \beta_{\mu 1}^{b_{\mu 1}} \beta_{\mu 2}^{b_{\mu 2}} \ldots ; \beta_{\mu\omega}^{b_{\mu\omega}} \ldots \beta_{\mu\lambda}^{b_{\mu\lambda}} \ldots \qquad (\lambda < \lambda_\mu)$$

$$= a_1 \beta_{11}^{b_{11}} \beta_{12}^{b_{12}} \ldots ; \beta_{1\omega}^{b_{1\omega}} \ldots \beta_{1\lambda}^{b_{1\lambda}} \ldots$$

$$+ a_2 \beta_{21}^{b_{21}} \beta_{22}^{b_{22}} \ldots ; \beta_{2\omega}^{b_{2\omega}} \ldots \beta_{2\lambda}^{b_{2\lambda}} \ldots \qquad (\lambda < \lambda_1 \text{ first line},$$

$$+ \ldots \qquad \lambda < \lambda_2 \text{ second line},$$

$$\ldots \qquad \text{and so on; } \mu < \alpha)$$

$$+ a_\omega \beta_{\omega 1}^{b_{\omega 1}} \beta_{\omega 2}^{b_{\omega 2}} \ldots ; \beta_{\omega\omega}^{b_{\omega\omega}} \ldots \beta_{\omega\lambda}^{b_{\omega\lambda}} \ldots$$

$$+ \ldots$$

$$\ldots$$

$$+ a_\mu \beta_{\mu 1}^{b_{\mu 1}} \beta_{\mu 2}^{b_{\mu 2}} \ldots ; \beta_{\mu\omega}^{b_{\mu\omega}} \ldots \beta_{\mu\lambda}^{b_{\mu\lambda}} \ldots$$

$$\ldots$$

where α is an ordinal (variable with σ), and the π_μ are product-forms of F satisfying the inequalities

$$\pi_1 > \pi_2 > \ldots ; > \pi_\omega > \ldots > \pi_\mu > \ldots , \qquad \mu < \alpha,$$

the coefficients a_μ being real numbers. It is the totality of all such forms σ which constitutes F.

Sum of Two Sum-Forms.—Let π be a product-form of F. The sum of two forms $a\pi$ and $a'\pi$, where a and a' are real numbers, is defined as the form $(a + a')\pi$; of two forms $a\pi$ and $a'\pi'$, where π' is a product-form of F and $\pi' < \pi$, as the sum-form $a\pi + a'\pi'$. In general, the sum $\sigma + \sigma'$ of two sum-forms σ and σ' of F is defined as the algebraic sum of all the summands of the two forms, the coefficients of the same product-form being summed. This algebraic sum may be written as a sum-form of F. The *negative*, or inverse with respect to addition, of a sum-form $\sigma = \sum_{\mu<\alpha} a_\mu \pi_\mu$ is the sum-form $-\sigma = \sum_{\mu<\alpha} - a_\mu \pi_\mu$. It is readily shown that the sum-forms of F constitute an abelian group with respect to addition.

Order of Two Sum-Forms.—A form $\sum_{\mu<\alpha} a_\mu \pi_\mu$, with leading coefficient $a_1 \neq 0$, will be said to be *negative* or *positive* (< 0 or > 0) according as a_1

is negative or positive. A form σ will be said to precede a form σ' if the difference $\sigma' - \sigma = \sigma' + (-\sigma)$ is positive. It may be shown that the forms of F are thus linearly ordered.

Product of Two Sum-Forms.—We now complete our definition of the operation of multiplication for forms of F. We introduce for this purpose the following notion of *infinite sum*. Let S be a set of forms of F. The *sum* of the forms of S will be said to exist if the product-forms in the sum-forms of S are in normal order when arranged in descending order and each product form occurs at most a finite number of times in all the sum-forms of S. This sum may then be written as a sum-form of F. Let π and π' be product-forms of F and a and a' real numbers. The product of the two forms $a\pi$ and $a'\pi'$ is taken to be the form $a''\pi''$, where $a'' = aa'$, and $\pi'' = \pi\pi'$. In general, the product $\sigma\sigma'$ of two sum-forms $\sigma = \sum_{\mu<\alpha} a_\mu \pi_\mu$ and $\sigma' = \sum_{\mu<\alpha'} a'_\mu \pi'_\mu$ is defined[2] as the sum of all possible products $a\pi a'\pi'$ where $a\pi$ is a term of σ and $a'\pi'$ a term of σ'. This sum may be shown to exist.

As previously stated the reciprocal of a product-form $\pi = \prod_{\lambda<\mu} \beta_\lambda^{b_\lambda}$ is the product-form $\pi^{-1} = \prod_{\lambda<\mu} \beta_\lambda^{-b_\lambda}$. We have, furthermore,

THEOREM 1. *The reciprocal*[3] *of a form* $\sigma = \sum_{\mu<\alpha} a_\mu \pi_\mu$ *with* $a_1 \neq 0$, *exists in* F *and is given by the expression* $a_1^{-1}\pi_1^{-1}(1 - \sigma' + \sigma'^2 - \sigma'^3 + \ldots \pm \sigma'^n \mp \ldots)$, *where* σ' *is determined by the relation* $\sigma = a_1\pi_1(1 + \sigma')$.

It follows readily that F is a field. We now define, in an abstract manner, *order-group* and *order-field*.[4] Later we define *closed order-field* and identify F as such a field.

Order-Group.—By an *order-group* we understand an abelian group G whose elements $\alpha, \beta, \ldots$ are linearly ordered in such a way that

1. For every element $\alpha \neq u$, where u is the unit of G, either $\alpha < u < \alpha^{-1}$ or $\alpha^{-1} < u < \alpha$ where α^{-1} is the inverse of α in G.
2. $\alpha > u$, $\beta > u$ implies $\alpha \cdot \beta > u$ where $\alpha \cdot \beta$ is the product of α and β in G.

Order-Field.—By an *order-field* we understand a field whose elements $\alpha, \beta, \ldots$ are linearly ordered in such a way that

1. For every element $\alpha \neq 0$, either $\alpha < 0 < -\alpha$ or $-\alpha < 0 < \alpha$.
2. $\alpha > 0$, $\beta > 0$ implies $\alpha + \beta > 0$ and $\alpha\beta > 0$.

It may be shown that the elements of an order-field form an order-group under the operation of addition; also the positive elements form an order-group under the operation of multiplication.

The notions *archimedean* and *relatively archimedean*, which we now introduce for order-fields, are fundamental for our consideration.[5]

Relatively Archimedean Elements.—Suppose, first, α and β are two elements of an order-field both $\geqq 1$. α and β will be said to be *relatively*

archimedean—archimedean with respect to each other—if each comes before some integral multiple or some integral power of the other. In general, two non-zero elements α and β will be said to have this relationship if α' and β' are relatively archimedean, where α' is one of the elements α, $-\alpha$, α^{-1}, $-\alpha^{-1}$ which is $\geqq 1$, and β' has a similar meaning. For convenience we say that zero is archimedean with respect to itself. It is clear that the archimedean relationship is reflexive, symmetric and transitive.

Archimedean Elements.—The positive integers, and the elements archimedean with respect to the positive integers, will be called *archimedean.* Elements not having this property will be called *non-archimedean.*

Archimedean Order-Fields.—An order-field every non-zero element of which is archimedean will be termed archimedean; otherwise it will be termed non-archimedean.

Real Extension.—If A is an order-field and B an order-field properly containing A as sub-order-field, B will be said to be a *real extension* of A.

Algebraic Real Extension.—We distinguish between an *algebraic real extension* and a *transcendental real extension* according as the field B is an algebraic or transcendental extension of A.

Archimedean Extension.—B will be said to be an *archimedean* extension of A if for every element β of B there exists an element α of A such that β is archimedean with respect to α; otherwise B will be said to be a *non-archimedean extension* of A.

THEOREM 2. *An algebraic real extension of an order-field is an archimedean extension.*

This theorem is a consequence of the following theorem which gives what may be deemed an "approximation" for the zeros of a polynomial equation in terms of the coefficients of the polynomial.

THEOREM 3. *If a positive element ζ of an order-field A is a zero of a polynomial* $\alpha_0 \xi^n + \alpha_1 \xi^{n-1} + \ldots + \alpha_n$, with coefficients in A, there exist distinct coefficients α_r and α_s such that $\left| \alpha_r/\alpha_s \right| \leqq \zeta^{r-s} \leqq n \left| \alpha_r/\alpha_s \right|$ (where $|\alpha_r/\alpha_s|$ is the positive one of the two values α_r/α_s and $-\alpha_r/\alpha_s$).

ξ-Extension.—If B is a real extension of an order-field A and consists of the rational expressions in a letter ξ with coefficients in A, we call B a *ξ-extension* of A. This ξ-extension is, of course, algebraic or transcendental according as ξ is or is not taken to satisfy a polynomial equation with coefficients in A. We note also that archimedean, transcendental extensions exist, and that a transcendental ξ-extension may be non-archimedean even if ξ is archimedean with respect to every element of A.

r-Algebraically Closed Order-Field.—An order-field for which no algebraic real extensions exist will be said *to be r-algebraically closed.*

Closed Order-Field.—An order-field for which no archimedean extensions exist we call *closed.* It follows from Theorem 2 that a closed order-field is

r-algebraically closed. The system F has the remarkable property that

THEOREM 4. *F is a closed order-field.*

Isomorphism.—Two order-fields are said to be *isomorphic* if there exists a bi-unique correspondence between them preserving order and field properties.

The real number system is characterized by the properties that it is a closed order-field and that every closed order-field contains a sub-order-field isomorphic with it. It is, except for isomorphism, the only *closed, archimedean order-field.* In view of this fact and Theorem 4, and the analogy of F with the system of decimal developments in respect to structure of element, addition, multiplication and order, we call the forms of F (other than the real numbers themselves) *transfinite real numbers.*

Existence and Uniqueness of Real Extensions.—Two ξ-extensions of an order-field are generally not isomorphic even if ξ is assumed to have the same position relative to the elements of A for both extensions. We have, however, the following theorem due to Artin and Schreier on the existence and uniqueness of algebraic real extensions.

THEOREM 5. (Artin and Schreier.) *Every order-field can be extended by means of algebraic real extensions to an r-algebraically closed order-field, and two such extensions are isomorphic, the isomorphism being unique when A corresponds to itself, element for element.*

The following theorem holds on the existence and uniqueness of transcendental ξ-extensions.

THEOREM 6. *If A is an r-algebraically closed order-field, and (A_1, A_2) a Dedekind cut of A, there exists a ξ-extension of A such that ξ follows every element of A_1 and precedes every element of A_2. In respect to isomorphism, this ξ-extension is unique.*

Theorem 6 is proved with the aid of the

LEMMA. *If A is an r-algebraically closed order-field, B a ξ-extension of A, $\overline{B}$ an r-algebraically closed, algebraic real extension of B, and $P(\bar{\xi})$ a polynomial in $\bar{\xi}$ with coefficients in A, where $\bar{\xi}$ is regarded as a variable element of $\overline{B}$, there exists an interval (α, β) of $\overline{B}$, with α, β elements of A and $\alpha < \xi < \beta$, such that $P(\bar{\xi})$ is properly monotone in (α, β)—i.e., is either always retrogressive or always progressive as $\bar{\xi}$ advances.*

With the aid of Theorems 4, 5 and 6, we establish closure and uniqueness properties of F, as formulated in Theorems 7, 8 and 9.

THEOREM 7. *If A is a sub-order-field of F and B an archimedean extension of A, then B is isomorphic with a sub-order-field of F under a correspondence in which A corresponds to itself, element for element.*

Base Order.—We have associated with every linear order L the closed order-field F containing L as sub-order. We find, furthermore, that there may be associated with every order-field A—whether closed or not—a sub-order L^* of A such that when A is isomorphic with F the order L^* is simi-

lar to the order L. This L^*, called the base order of A, is defined as follows: A sub-order L^* of an order-field A consisting of non-archimedean elements of A is said to be a *base order* of A if no two elements of L^* are archimedean with respect to each other and every non-archimedean element of A is archimedean with respect to some element of L^*.

THEOREM 8. *Every order-field of base order similar to L is isomorphic with a sub-order-field of F.*

THEOREM 9. *F is the only closed order-field of base order L, except for isomorphisms.*

F may thus be considered as the "maximal" order-field having L as base order. When L is the null set, F is the real number system. We now obtain an order-field R which may be considered as the "minimal" order-field of base order L. First, we define

Transfinite Integer.—The forms belonging to F of type

$$p_{11}\beta_{11}^{q_{11}}\beta_{12}^{q_{12}} \ldots \beta_{1n_1}^{q_{1n_1}} + p_{21}\beta_{21}^{q_{21}}\beta_{22}^{q_{22}} \ldots \beta_{2n_2}^{q_{2n_2}} + \ldots + p_{m_1}\beta_{m_1}^{q_{m_1}}\beta_{m_2}^{q_{m_2}} \ldots \beta_{mn_m}^{q_{mn_m}}$$

where the p's are integers, and m, the q's and the n's positive integers, form a domain of integrity I and are called (with the exception of the integers themselves) *transfinite integers.*

Transfinite Rational Number.—We call the forms of F expressible as the quotient of two forms of I (with the exception of the rational numbers themselves) *transfinite rational numbers.* It may be shown that the subsystem of F consisting of the rational numbers and the transfinite rational numbers form an order-field R of base order L.

THEOREM 10. *Every order-field of base order L contains an order-field isomorphic with R.*

Thus R is "minimal," and a generalization, in this sense, of the rational numbers.

Transfinite Complex Number.—The forms $\alpha + i\beta$ where α and β are forms of F and $i^2 = -1$, are called (except for the complex numbers themselves) *transfinite complex numbers.* Artin and Schreier have shown that if A is an r-algebraically closed order-field, the field $A(i)$ whose elements are the forms $\alpha + i\beta$, where α and β are elements of A, has algebraic closure. Therefore, $F(i)$ has algebraic closure and we may conclude[6] that

THEOREM 11. *Every field of characteristic zero of cardinal not exceeding the cardinal of F(i) is isomorphic with a subfield of F(i).*

Some other notions for the real number system may be extended to F; for example, *prime, sine, power series, limit of a sequence, metric space.*

The present paper will be offered *in extenso* to the *Transactions* of the American Mathematical Society.

[1] I wish to express my profound gratitude to Professor H. Blumberg for his generous aid in my preparation of the present paper and the inspiration furnished through his courses at Ohio State University and my close companionship with him.

[3] Cf. Hausdorff, *Mengenlehre* (1914), Chap. VI, pp. 194–209, where too, a system whose general element is a structure which may be regarded as a generalization of decimal development is devised and *product* of two structures is defined in a manner analogous to ours.

[3] We define $(1 + \alpha)^a$ where a is a real number and α a form of F preceding 1, as the sum $1 + a\alpha + a(a - 1)/2!\alpha^2 + \ldots$. This sum may be written as a form of F.

[4] Artin and Schreier, *Algebraische Konstruction reeller Körper*, Abh. des Math. Sem., Hamburg, Vol. 5, 1926.

[5] The meaning we attach to *relatively archimedean* differs essentially from that of Artin and Schreier.

[6] We may define the *modular field* $F(p)$ corresponding to the prime integer p as the field consisting of the forms

$$\sum_{\mu<\alpha} p_\mu \pi_\mu = p_1\pi_1 + p_2\pi_2 + \ldots; + p_\omega\pi_\omega + \ldots p_\lambda\pi_\lambda + \ldots, \quad \mu < \alpha,$$

where the π_μ are product-forms of F and the p_λ are elements of the field which is the algebraically closed algebraic extension of the field whose elements are the integers modulo p. The writer has grounds for surmising that every field of characteristic p is isomorphic with a subfield of $F(p)$.

GROUPS WHICH CONTAIN A HAMILTONIAN SUBGROUP OF ODD PRIME INDEX

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The groups which contain an abelian subgroup of odd prime index were considered in an earlier number of these PROCEEDINGS[1] and it was proved there that when this index is p and the subgroup is not invariant then the group G contains also an invariant subgroup of order p and is the direct product of a group whose order is a power of p and some other group whenever its order is divisible by p^2. Moreover, G contains p conjugated subgroups of index p which are permuted under G according to the metacycle group or order $p(p - 1)$ or according to an invariant subgroup thereof. When G is not a direct product the quotient group with respect to its invariant subgroup of order p involves only operators whose orders divide a power of $p - 1$ and is cyclic with respect to the central of G. The last seven words of this sentence were unfortunately omitted from the statement of this theorem in the article noted above as was pointed out in a letter to me by Helmut Wielandt.

When G contains a non-invariant Hamiltonian subgroup of index p each of these subgroups is again transformed into itself by only its own operators. The common operators of any two of these subgroups generate

separately invariant cyclic subgroups such that the operators of these cyclic subgroups are either invariant under G or have just two conjugates thereunder. Hence the p conjugate Hamiltonian subgroups of index p under G are again transformed by G according to a transitive group of degree p. The subgroup of this transitive group which corresponds to the cross-cut of the given p Hamiltonian subgroups of index p is the identity since a transitive group of degree p cannot contain an invariant subgroup composed of permutations of order 2 in addition to the identity. The p conjugate Hamiltonian subgroups of index p are therefore transformed by G according to a transitive group of class $p - 1$. This group is therefore either the metacyclic group of order $p(p - 1)$ or a transitive subgroup thereof whose order exceeds p.

The operators of G which correspond to the subgroup of order p in this quotient group are relatively commutative and hence the Sylow subgroup of G whose order is a power of p is abelian. Hence G contains a power of p operators of lowest order which correspond to an operator of order p in this quotient group. At least one of these is transformed into a power of itself by an operator of G which corresponds to an operator in the given quotient group whose order divides $p - 1$. As this operator is commutative with each of the operators of G which are common to its p conjugate Hamiltonian subgroups of index p it results that G contains an invariant subgroup of order p and is a direct product whenever its order is divisible by p^2. This proves the following theorem: *If a group contains a non-invariant Hamiltonian subgroup of odd prime index* p *the operators which are common to the* p *conjugates of this subgroup appear in an abelian subgroup of the group and the group contains an invariant subgroup of order* p. *If the order of the group is divisible by* p^2 *it is a direct product.*

The fact that there is at least one group which contains non-invariant subgroups of index p which are transformed thereunder according to an arbitrary transitive substitution group of degree p whose order exceeds p results directly from the theorem that the p subgroups of any such transitive group of degree p which are separately composed of all the substitutions which omit a given letter are transformed under the group in the same way as its letters are permuted thereunder. That is, the cyclic group of degree p, p being a prime number, is the only transitive group of degree p which is not the group of permutations of some non-invariant subgroup of index p when this subgroup is transformed by all the operators of the group. It, therefore, results that the manner in which p conjugate subgroups of index p can be transformed under the group may vary widely when the properties of these subgroups are not fixed.

Suppose now that the p conjugate subgroups of index p under G are quaternion. Two of these subgroups have an operator of order 4 in common since the quotient group with respect to the group generated by these

common operators and an operator of order p is cyclic. In fact this quotient group is cyclic in the more general case when the p conjugate subgroups of index p under G are any Hamiltonian group. It therefore results that when these p conjugate subgroups are the quaternion group then G contains the cyclic group of order $4p$ as a subgroup of index 2 and it is therefore the dicyclic group of order $8p$. When these p conjugate subgroups are abelian they do not necessarily have any common operator besides the identity but when they are Hamiltonian they always have common operators besides the identity. In the latter case these common operators always include operators of order 4.

Every Hamiltonian group whose order exceeds 8 is the direct product of the quaternion group and an abelian group. When G contains a non-invariant Hamiltonian group of index p this abelian group appears in its central since the quotient group with respect to its subgroup corresponding to the subgroup of order p in its group of transformation is cyclic. Hence there results the following theorem: *If a group contains a Hamiltonian group as a subgroup of odd prime index then it is the direct product of an abelian group and the dicyclic group of order* 8p, p *being an odd prime number.* By means of this theorem it is easy to determine all the groups of a given order which involve a non-invariant Hamiltonian subgroup of index p. When p is given the number of these groups is the number of the abelian groups whose order is equal to the quotient of the given order divided by $8p$.

The main objects of the present note are to supplement the article to which reference was made in the first paragraph of this note and to correct the omission to which attention was called near the close of this paragraph. It seemed desirable to try to clarify several statements contained in the given article in order to avoid possible misunderstanding in regard to their meanings. It may be added that this article is closely related to one by O. Hölder, published in the *Mathematische Annalen*, volume 46, pages 321–422 (1895). Comparatively little progress has been made since then in the determination of groups when the invariant subgroup and the corresponding quotient group are given.

[1] G. A. Miller, *Proc. Nat. Acad. Sci.*, **23**, 13–16 (1937).

DIFFERENTIAL INVARIANTS IN A GENERAL DIFFERENTIAL GEOMETRY[1]

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1. *Introduction.*—The study of general differential geometries in which the "coördinate space" is a Banach space[2] instead of the traditional n-dimensional arithmetic space has been initiated recently by one of us.[3] The theories include as special cases not only the finite dimensional but also the infinite dimensional Riemannian and non-Riemannian geometries.[4] Normal coördinate systems are even more important in the infinite dimensional than in the finite dimensional geometries, on account of the more formidable nature of the calculations involved. A theory of normal coördinates for a general differential geometry with a linear connection has been developed by us and applied to the study of fundamental sets of differential invariants.[5] Many aspects of these developments are novel even if the Banach space is specialized to be a finite dimensional arithmetic space.

In this note we indicate how the theory of normal coördinates may be extended in order to handle differential invariants involving explicitly covariant[6] as well as contravariant vectors. This necessitates the study of new types of "adjoint" functional equations, which are of some interest in themselves, aside from their use in general differential geometry. The complete developments with proofs will appear elsewhere.[7]

2. *An Existence Theorem for an Abstract Functional Equation.*—Let E be a Banach space with an independently postulated real symmetric bilinear functional $[x, y]$ such that $[x, y] = 0$ for all $x \epsilon E$ implies $y = 0$. Let $\Gamma(x, \xi_1, \xi_2)$ be bilinear in ξ_1, ξ_2 where $x, \xi_1, \xi_2, \Gamma \epsilon E$. Consider the differential system

$$\frac{d^2x}{ds^2} + \Gamma\left(x, \frac{dx}{ds}, \frac{dx}{ds}\right) = 0, \; x(0) = p, \left(\frac{dx}{ds}\right)_0 = \xi \qquad (1)$$

whose solution[8] is $x = \mu(s\xi, p)$. The "variational system" of (1) is[9]

$$\left.\begin{aligned} \frac{d^2z}{ds^2} + \Gamma(\mu, \mu', \mu'; z) + 2\Gamma\left(\mu, \frac{dz}{ds}, \mu'\right) = 0 \\ z(0, \lambda) = \lambda, \left(\frac{dz(s, \lambda)}{ds}\right)_{s=0} = -\Gamma(p, \lambda, \xi), \end{aligned}\right\} \qquad (2)$$

where $\mu' = \dfrac{d\mu(s\xi, p)}{ds}$, while the "adjoint variational system" of (1) is[10]

$$\left.\begin{aligned} &\frac{d^2z^*}{ds^2} + z^*(s, \Gamma_{(4)}{}^*(\mu, \mu', \mu'; \lambda)) \\ &+ 2\left[\frac{d}{d\sigma} z^*(\sigma, \Gamma_{(2)}{}^*(\mu, \lambda, \mu'))\right]_{\sigma = s} = 0 \\ &z^*(0, \lambda) = \lambda, \left(\frac{dz^*(s, \lambda)}{ds}\right)_0 = -\Gamma_{(2)}{}^*(p, \lambda, \xi). \end{aligned}\right\} \quad (3)$$

The following fundamental existence theorem for system (3) has been proved by us using a variety of methods and results in general analysis.[11]

THEOREM 1. *Let* X *be a neighborhood of a point* x_0 *of the space* E *and let* $\Gamma(x, \xi, \eta$ *be a function on* XE^2 *to* E *bilinear and symmetric in* ξ, η *and satisfying the following conditions:*

(i) $\Gamma(x, \xi, \eta)$ *is of class* $C^{(n)}$ *uniformly on*[12] $XE^2((0)_1)$;

(ii) *The adjoint* $\Gamma_{(2)}{}^*(x, \xi, \eta)$ *exists and is of class* $C^{(n)}$ *uniformly on* $XE^2((0)_1)$;

(iii) *The adjoint* $\Gamma_{(4)}{}^*(x, \xi, \eta; \lambda)$ *exists and is of class* $C^{(n-1)}$ *uniformly on* $XE^3((0)_1)$.

Let Y_0X_0 *be the* ξ, p *range of definition of the solution (known to exist by Theorem* 1.1 *of* MH) *of differential system* (1).

Then the "adjoint differential system" (3) *has a unique solution* $z^* = \psi(s, \xi, p, \lambda)$, *which is of class* $C^{(n-1)}$ *in* ξ, p, λ *uniformly on* $Y_0X_0E((0)_1)$, *for each* s *in* I, *where* I *is the interval* $0 \leq s \leq 1$. *Furthermore, the adjoint* $\mu_{(3)}{}^*(s\xi, p; \lambda)$ *of the solution* $\mu(s\,\xi, p; \lambda)$ *of the differential system* (2) *exists equal to* $\psi(s, \xi, p, \lambda)$.

3. *Normal Coördinate Systems of Class* $k^{(n)}$.—In a general differential geometry with a linear connection $\Gamma(x, \xi_1, \xi_2)$ the equation of paths takes the form of the differential equation in system (1) where s is an affine parameter. A coördinate system $y(P)$ in which the equation of any path through a point P_0 of the "geometric" space with coördinate $y = 0$ takes the form $y = s\xi$, where ξ is a constant element of E, will be called a *normal coördinate system.* Before stating our existence theorem for normal coordinate systems it is necessary to give the

Definition.—A biunivocal transformation $\bar{x} = \bar{x}(x)$ carrying an open set $S_1 \subset E$ into an open set $S_2 \subset E$ will be said to be of *class* $k^{(m)}$ if it satisfies the following conditions:

(i) The function $\bar{x}(x)$ and its inverse $x(\bar{x})$ are of class $C^{(m)}$ locally uniformly[13] at each point of their domains S_1 and S_2;

(ii) The adjoint $\bar{x}^*(x; \delta x)$ exists, is a solvable linear function of δx and is of class $C^{(m-1)}$ locally uniformly at each $x \epsilon S_1$.

The postulates for the geometry[13] will be taken as in MH, with transformations of class $K^{(m)}$ replaced by those of the more restrictive class $k^{(m)}$. The following key theorem may be proved with the aid of Theorem 1.

THEOREM 2. *Let* q *be any chosen point of the coördinate domain* Σ *of an allowable* $k^{(n+2)}$ *coördinate system* x(P) *in which the following conditions are satisfied by the linear connection* $\Gamma(x, \xi_1, \xi_2)$ *for each point* $x \epsilon \Sigma$:

(i) $\Gamma(x, \xi_1, \xi_2)$ *is a symmetric bilinear function of* ξ_1 *and* ξ_2;

(ii) $\Gamma(x, \xi_1, \xi_2)$ *is of class* $C^{(n)}$ *locally uniformly at* x;

(iii) *The adjoint* $\Gamma_2^*(x, \xi_1, \xi_2)$ *exists and is of class* $C^{(n)}$ *locally uniformly at* x;

(iv) *The adjoint* $\Gamma_{(4)}^*(x, \xi_1, \xi_2; \lambda)$ *exists and is of class* $C^{(n-1)}$ *locally uniformly at* x.

Then there exists a constant $d > 0$ *and a function* h(p, x) *of class* $C^{(n)}$ *uniformly on* $E^2((q)_{2d})$ *such that for any choice of* p *in* $E((q)_d)$ *the transformation* $y = h(p, x)$ *with inverse* $x = \mu(y, p)$ *is of class* $k^{(n)}$ *for* $x \epsilon E((p)_d)$ *and defines a normal coördinate system* y(P) *with center* $P_0 = P(p)$.

4. *Differential Invariants.*—With the aid of the above existence theorem we have extended the differential invariant theory[14] of MH to multilinear forms in both covariant and contravariant vectors whose values are either scalar, covariant vector or contravariant vector fields. Finally, we have shown that every differential invariant (of a certain general type) whose components are functionals of the linear connection Γ and its adjoint, the differentials of Γ and their adjoints, can be written by a simple replacement process in terms of a fundamental set of differential invariants, namely, the normal vector forms and their adjoints.

[1] Presented to the American Mathematical Society, September, 1936, and November, 1936. See abstracts 348 and 456, *Bull. Amer. Math. Soc.*, **42** (1936).

[2] Banach, S., *Théorie des Opérations Linéaires*, Warsaw (1932).

[3] Michal, A. D., "Abstract Covariant Vector Fields in a General Absolute Calculus," *Amer. Journal of Math.*, **59**, 306–314 (1937); "Postulates for Linear Connections in Abstract Vector Spaces," *Annali di Matematica*, **15**, 197–220 (1936); "General Tensor Analysis," *Bull. Amer. Math. Soc.*, **43**, 394–401 (1937); "Riemannian Differential Geometry in Abstract Spaces," these PROCEEDINGS, **21**, 526–529 (Sept., 1935); several abstracts in *Bull. Amer. Math. Soc.*, **39–43** (1933–1937) and several forthcoming papers.

[4] Michal, A. D., "Affinely Connected Function Space Manifolds," *Amer. Jour. Math.*, **50**, 473–517 (1928); "Differential Geometries of Function Space," these PROCEEDINGS, **16**, 88–94 (Jan., 1930); "The Differential Geometry of a Continuous Infinitude of Contravariant Functional Vectors," *Ibid.*, **16**, 162–164 (1930); "Projective Functional Tensors and Other Allied Functionals," *Ibid.*, **16**, 165–168 (1930); "Function Space Time Manifolds," *Ibid.*, **17**, 217–225 (1931).

[5] Michal, A. D., and Hyers, D. H., "Theory and Applications of Abstract Normal Coördinates in a General Differential Geometry," *Tonelli's Annali di Pisa* (in the hands of the editors). *We shall refer to this paper as* MH.

[6] See first paper in footnote 3.

[7] Michal, A. D., and Hyers, D. H., "Differential Invariants in a General Differential Geometry," *Mathematische Annalen* (in the hands of the editors).

[8] MH, sections 1 and 3.

[9] By $f(x, y_1, \ldots, y_n; \lambda)$ we shall mean the Fréchet differential of $f(x, y_1, \ldots, y_n)$ in x with λ as increment.

[10] If $f(x_1, \ldots, x_n)$ is linear in the ith place, then by $f^*_{(i)}(x_1, \ldots, x_n)$ we shall mean the adjoint of $f(x_1, \ldots, x_n)$ considered as a linear function of x_i.

[11] See the following two papers and the numerous references given there: Michal, A. D., and Hyers, D. H., "Second Order Differential Equations with Two Point Boundary Conditions in General Analysis," *Amer. Jour. Math.*, **58**, 646–660 (1936); Michal, A. D., and Elconin, V., "Completely Integrable Differential Equations in Abstract Spaces," *Acta Mathematica*, **68**, 71–108 (1937).

[13] For the notion of "function of class $C^{(n)}$" see Hildebrandt, T. H., and Graves, L. M., "Implicit Functions and Their Differentials in General Analysis," *Trans. Amer. Math. Soc.*, **129**, 127–153 (1927). By $E((x_0)_a)$ we mean the set of points $x \epsilon E$ such that $\| x - x_0 \| < a$.

[13] MH, section 2.

[14] MH, sections 5–7.

ERRATA

In "Continuous Rings and Their Arithmetics," by J. v. Neumann, *Proc. Nat. Acad. Sci.*, June, 1937, **23**, 341–349:

Due to an oversight, a line was omitted on p. 347, destroying the sense of paragraph 8. It should be corrected as follows: Insert between lines 12 and 11 from the bottom of page 347: "*integer* if $p(a) = 0$ for a suitable integer polynomial $p(x)$. An $a \epsilon \mathfrak{R}$ is a *general*"